Directed percolation depinning models: Evolution equations

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We present the microscopic equation for the growing interface with quenched noise for the model first presented by Buldyrev *et al.* [Phys. Rev. A **45**, R8313 (1992)]. The evolution equation for the height, the mean height, and the roughness are reached in a simple way. The microscopic equation allows us to express these equations in two contributions: the contact and the local one. We compare these two contributions with the ones obtained for the Tang and Leschhorn model [Phys. Rev A **45**, R8309 (1992)] by Braunstein *et al.* [Physica A, **266**, 308 (1999)]. Even when the microscopic mechanisms are quiet different in both models, the two contributions are qualitatively similar. An interesting result is that the diffusion contribution, in the Tang and Leschhorn model, and the contact one, in the Buldyrev model, leads to an increase of the roughness near the criticality. [S1063-651X(99)00504-8]

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I. INTRODUCTION

The description of the noise-driven growth on a self-affine interface far from equilibrium is a challenging problem. The interface has been characterized through scaling of the interfacial width *w* with time *t* and lateral size *L*. The result is the determination of two exponents β and α called dynamical and roughness exponents, respectively. It is well known that interfacial width $w \sim L^{\alpha}$ for $t \gg t^*$ and $w \sim t^{\beta}$ for $t \ll t^*$, where $t^* \simeq L^{\alpha/\beta}$ is the saturation time. These properties occur for many models of surface growth. The values of the exponents leads to the classification of these models in different universality classes. Several models, belonging to the same directed percolation depinning (DPD) universality class, have been introduced to explain experiments on fluid imbibition in porous media, roughening in slow combustion of paper, growth of bacterial colonies, etc.

It is currently accepted that the quenched disorder plays an essential role in those experiments. The DPD models take into account the most important features of the experiments [1,3]. The two first models were simultaneously introduced by Buldyrev *et al.* [1] and Tang and Leschhorn [2] to explain the fluid imbibition in paper sheet. Several authors have focused their attention on scaling properties and relationships between the dynamical and statics exponent for these models. The Tang and Leschhorn (TL) model has recently reviewed by Braunstein *et al.* [4,5] from a different point of view than the traditional one. The principal contribution was the restatement of the microscopic equation (ME) for the TL model (see Appendix). This equation allows the separation into two contributions: the substratum contribution by local growth and the diffusion one. They found that the diffusion contribution to the temporal derivative of the square roughness may be either negative or positive and that the behavior of this contribution depends on the pressure *p*. The negative contribution tends to smooth out the surface; this case dominates for $p > p_c$ (where $p_c = 0.461$ is the critical pressure). The positive contribution enhances the roughness. At the critical pressure the substratum contribution to the temporal derivative of the square roughness is practically constant, but the diffusion contribution is very strong. This last contribution has important duties on the power law behavior.

In this paper we focus the attention in the Buldyrev *et al.* model of DPD. We show that this model presents several qualitative features of the TL model. We write a ME, starting from the microscopic rules, for the evolution of the height as a function of time. The ME allows us the identification of two contributions that dominates the dynamics of the system, the ''contact'' and the ''local'' one. In this context we study the mean height speed (MHS) and the temporal derivative of the square interface width (DSIW) as a function of time. We show that the contact contribution smooths out the surface for *p* well above the criticality but enhances the roughness near the critical value. To our knowledge, the separation into two contributions for all the quantities studied in the present paper and the important duties of the contact contribution to the critical power law behavior has never been studied before. The paper is organized as follows. In Sec. II we write the microscopic equation for the evolution of height, the mean height, and the roughness for the Buldyrev *et al.* model. We study the MHS, analyzing the contact and the local contributions. Also, the two contributions to the DSIW are analyzed. This separation into two contributions allows us to explain the mechanism of roughening. In Sec. III we compare the Buldyrev *et al.* model with the TL model. Finally, we conclude with a discussion in Sec. IV.

II. BULDYREV *et al.* MODEL

A. Microscopic equation

The interface growth takes place in a lattice of edge *L* with periodic boundary conditions. A random pinning force $g(\mathbf{r})$ uniformly distributed in [0,1] is assign to every cell of

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the lattice. For a given pressure p, the cells are divided in two groups, active (free) cells with $g(\mathbf{r})\leqslant p$ and inactive (blocked) cells with $g(\mathbf{r})>p$. The interface between wet and dry cells is specified by a set of integer column heights h_i ($i=1,\ldots,L$). At $t=0$ we start with flat initial conditions, i.e., $h_i=0$ for all i. During the growth, a column is selected at random with probability $1/L$ and the highest dry active cell, in the chosen column, that is nearest neighbor to a wet cell is wetted. Afterwards, we wet all the dry cells below it. In this model, the time unit is defined as one growth attempt. In numerical simulations at each growth attempt, the time t is increased by $\delta t=1/L$. In this way, after L growth attempts, the time is increased by one unit. In our simulations we use $L=8192$.

We consider the evolution for the height of the ith site of the process described above. Let us denote by $h_i(t)$ the height of the i-th generic site at time t. From the microscopic rules we obtain the evolution for the ith height in the next time step $\delta t=1/L$,

$$h_i(t+\delta t)=h_i(t)+\delta t\{\Theta(-z_i)F_i(h_i+1)+[1-\Theta(-z_i)]Y_i\}, \tag{1}$$

where $\Theta(x)$ is the unit step function defined as $\Theta(x)=1$ for $x\geqslant 0$ and equals to 0 otherwise, $F_i(h_i+j)$ equal to 1 if the cell is active and 0 if the cell is inactive $(1\leqslant j\leqslant z_i)$,$z_i=\max(h_{i-1},h_{i+1})-h_i$, and

$$\begin{aligned}Y_i=&z_iF_i(h_i+z_i)+(z_i-1)F_i(h_i+z_i-1)\\&\times[1-F_i(h_i+z_i)]+(z_i-2)F_i(h_i+z_i-2)\\&\times[1-F_i(h_i+z_i-1)][1-F_i(h_i+z_i)]+\cdots\\&+F_i(h_i+1)[1-F_i(h_i+2)]\cdots\\&\times[1-F_i(h_i+z_i-1)][1-F_i(h_i+z_i)],\end{aligned}$$

is the increase of the height in the ith column due to the contact with the nearest lateral neighbor. The term between braces in Eq. (1) takes into account all the possible ways the site i can growth. The height in the site i is increased by (i) 1 if $h_i\geqslant\min(h_{i+1},h_{i-1})$ and $F_i(h_i+1)=1$, or (ii) Y_i if $h_i<\min(h_{i+1},h_{i-1})$ and $F_i(h_i+Y_i)=1$. Otherwise, the height is not increased. We shall call contact contribution to the term $[1-\Theta(-z_i)]Y_i$ [related to case (ii)] and local contribution to the term $\Theta(-z_i)F_i(h_i+1)$ [related to case (i)].

Averaging Eq. (1) over the lattice, taking $\delta t\to 0$, the evolution equation for the mean height is

$$\frac{dh}{dt}=\langle\Theta(-z_i)F_i\rangle+\langle[1-\Theta(-z_i)]Y_i\rangle, \tag{2}$$

and the evolution equation for the square interface width is

$$\frac{dw^2}{dt}=2\langle(h_i-\langle h_i\rangle)\Theta(-z_i)F_i\rangle+2\langle(h_i-\langle h_i\rangle)[1-\Theta(-z_i)]Y_i\rangle. \tag{3}$$

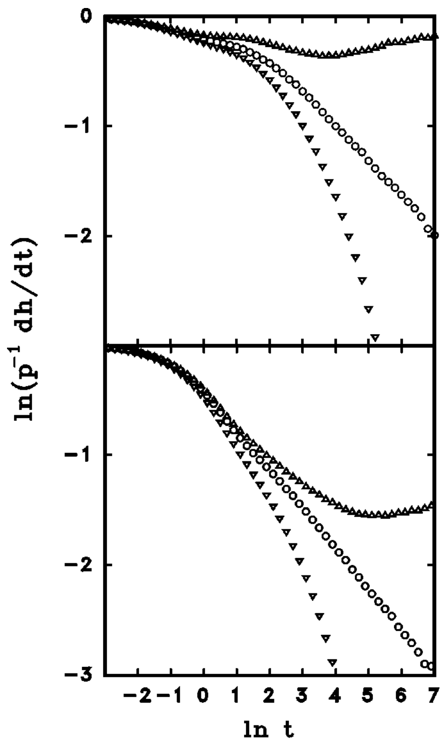

FIG. 1. Plots of $p^{-1}dh/dt$ vs t. The top (bottom) plot shows the results for the Buldyrev *et al.* (Tang and Leschhorn) model. For the top plot the parameter p is 0.56 (△), 0.531 (○), and 0.51 (▽). For the bottom plot the parameter p is 0.49 (△), 0.461 (○), and 0.4 (▽).

The first terms of both equations can be identified as the local growth contribution, and the second term as the contact growth contribution. We shall see in Sec. II C that the separation into these two analytical terms allows us to show how the contact mechanism enhances the roughness near the criticality. In the present paper we focus only on the dynamical behavior, i.e., $t\ll t^*\simeq L$ for the mean height and roughness (in the DPD models $\alpha/\beta=1$).

B. Mean height

The top plot of Fig. 1 shows the MHS as a function of time in three regimes (moving, critical, and pinning phases). The initial condition for the MHS is p in all regimes. At the criticality we found for the mean height a power law behavior with the same dynamical exponent that the roughness one $\beta=0.68\pm0.02$ for $p_c=0.531$. In the moving and pinning phase we can see that this power law does not hold. Below the criticality, in the pinning phase, the MHS goes to zero. Above, in the moving phase, the MHS goes to a certain constant value. The left plots of Fig. 2 show the contributions to the MHS: the local one $\langle\Theta(-z_i)F_i\rangle$ and the contact one $\langle[1-\Theta(-z_i)]Y_i\rangle$. The local contribution, which is equal to p at $t=0$, is stronger in the early time regime. This is because in this regime the difference of heights between nearest neighbors is mostly less than one. This contribution set into motion the contact growth. In the moving phase (see

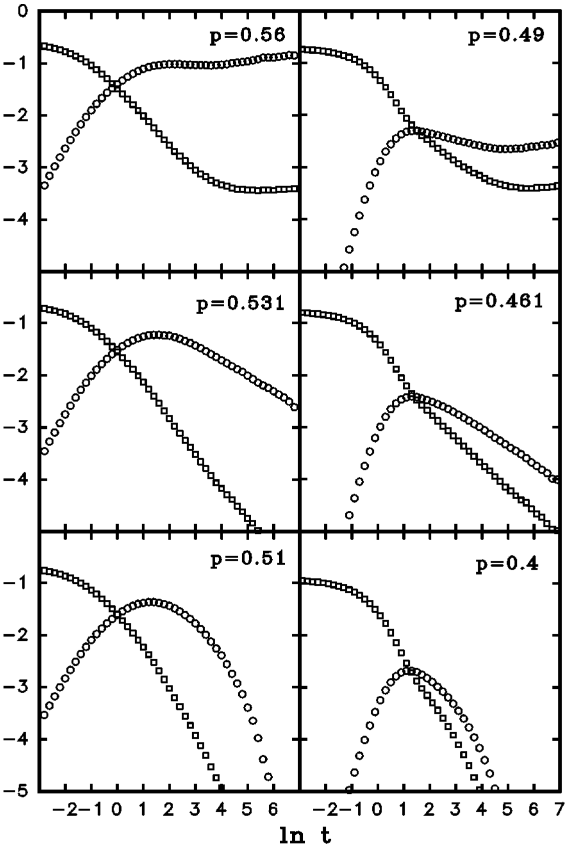

FIG. 2. ln-ln plots of different contributions to the MHS as function of time for the Buldyrev *et al.* model (left plots) and the Tang and Leschhorn model (right plots), for different values of p. The circles represent the contact contribution (left plots) and the diffusion contribution (right plots). The squares represent the local contribution (left plots) and the substratum contribution (right plots). For both models, the top, bottom, and middle plots show the moving, the pinning, and the critical phases, respectively.

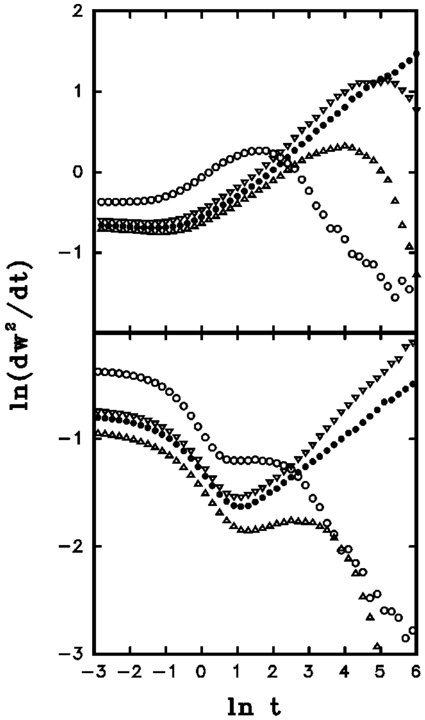

FIG. 3. DSIW as a function of time. The top (bottom) plot shows the results for the Buldyrev *et al.* (Tang and Leschhorn) model. For the top plot the parameter p is 0.7 ($\bigcirc$), 0.56 ($\bigtriangledown$), 0.531 ($\bullet$), and 0.51 ($\triangle$). For the bottom plot the parameter p is 0.7 ($\bigcirc$), 0.49 ($\bigtriangledown$), 0.461 ($\bullet$), and 0.4 ($\triangle$). In both models the symbol $\bullet$ shows the critical behavior.

left top plot of Fig. 2) both contributions go to a certain constant. In the intermediate regime the local contribution decreases while the contact one increases. At the criticality and in the pinning phase (see left middle and left bottom plots of Fig. 2, respectively), the local contribution decreases continuously from p to zero. The contact contribution increases from zero to a maximum value and then decreases reaching asymptotically the MHS. In all phases both contributions are equal at $t \simeq 1$. This means that after L-growth attempts, the interplay between both mechanisms are equal independently of p. After this, the dynamical behavior is strongly dominated by the contact mechanism.

C. Roughness

The top plot of Fig. 3 shows the temporal DSIW as a function of time for various values of p. The initial condition is p in all regimes. As we expected [6], the power law holds only at the criticality. The DSIW goes asymptotically to zero at the pinning and moving phase. In the left plots of Fig. 4, we show the two contributions to the DSIW for different values of p. The local contribution $2\langle(h_i - \langle h_i \rangle)\Theta(-z_i)F_i\rangle$ to the DSIW is always positive. As p decreases, this contribution becomes less important, but always rough the interface. On the other hand, for $p > p_c$, the contact contribution $2\langle(h_i - \langle h_i \rangle)[1-\Theta(-z_i)]Y_i\rangle$ can take negative values, smoothing out the surface. Otherwise, for $p \simeq p_c$, the contact contribution is always positive roughening the interface. One could expect that the contact contribution always smooths out the surface because it tends to widen the roughen picks. However, near the criticality, the contact growth happens mainly in lateral neighbors cells to few height terraces above the mean height. Then, this new wetted column smooths out locally, but it moves away from the mean height increasing the roughness.

III. COMPARISONS WITH THE TANG AND LESCHHORN MODEL

We rescue the similarities between the Buldyrev *et al.* and the Tang and Leschhorn models despite the strong microscopic differences between their rules. In a previous paper Braunstein *et al.* [4,5] wrote the ME for the TL model. They identified two separate contributions: the substratum and the diffusion one in the MHS and the DSIW (see Appendix). In the present paper, we also obtain two contributions: the local and the contact one. Figure 2 shows the contributions to the MHS for both models. Notice that each pair of plots have the same qualitative behavior. The shape of the diffusion and

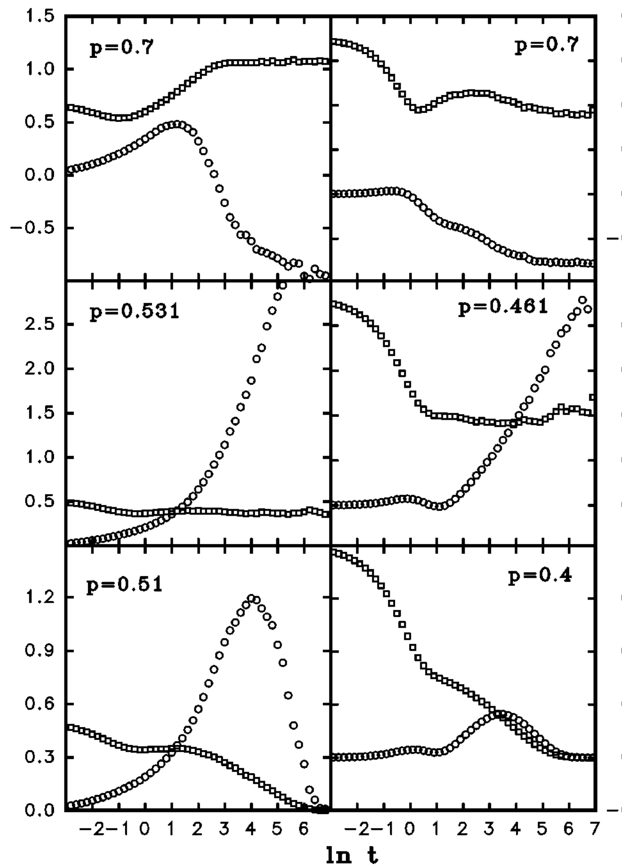

FIG. 4. Semi-ln plots of different contributions to the DSIW as function of time for the Buldyrev *et al.* model (left plots) and the Tang and Leschhorn model (right plots), for different values of p. The circles represent the contact contribution (left plots) and the diffusion contribution (right plots). The squares represent the local contribution (left plots) and the substratum contribution (right plots). For both models, the top, bottom, and middle plots show the moving, the pinning, and the critical regime, respectively.

substratum contribution in the TL model are qualitatively similar with the contact and local contribution in the Buldyrev *et al.* model, respectively, even when the microscopic processes are quite different for each model. The different contributions to the DSIW for both models are shown in Fig. 4. Notice that the diffusion and the contact contributions, in each model, can take negative values for $p > p_c$ smoothing out the interface. Near the criticality, in both models, the roughness is mainly due to the diffusion and the contact contributions. These last contributions play a very important role at the criticality in each model. The similarities between the models could explain why these two different microscopic models belong to the same universality class. Figure 3 shows the DSIW as a function of time for both models. As we expected [6], the power law holds only at the criticality despite other authors [1,2,7–9]. From these plots it is easy to see that this last statement holds for both models.

IV. CONCLUSIONS

We wrote the ME for the evolution of the height in the Buldyrev *et al.* model and we compared the results obtained with those from the TL model. Using the ME we studied the evolution of the mean height and the roughness. The ME allows us to separate the local and the contact contributions. We found that the contact contribution near the criticality is the main responsibility of the roughness. We found qualitatively that the shape of the contact contribution is analogous to that of the diffusion one in the TL model, and that the shape of the local growth is similar to that of the substratum contribution in the TL model. We found that the power law behavior holds only at the criticality for the Buldyrev *et al.* model. This last feature, common to the TL model, suggests to us that it may be common to all other DPD growth models [9].

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APPENDIX: MICROSCOPIC EQUATION FOR THE TANG AND LESCHHORN MODEL

We present here the microscopic equation for the TL model [4,5]. The time evolution equation for the interface height, in a time step δt, is

$$h_i(t+\delta t)=h_i(t)+\delta t[W_{i+1}+W_{i-1}+F_i(h_i')W_i], \tag{A1}$$

with

$$W_i=1-\Theta[h_i-\min(h_{i-1},h_{i+1})-2],$$

$$W_{i\pm1}=\Theta[h_{i\pm1}-\min(h_i,h_{i\pm2})-2]\times\{[1-\Theta(h_i-h_{i\pm2})]+\tfrac{1}{2}\delta_{h_i,h_{i\pm2}}\}.$$

where $h_i'=h_i+1$. Here $\Theta(x)$ is the unit step function defined as $\Theta(x)=1$ for $x\geqslant 0$ and equals to 0 otherwise. $F_i(h_i')$ equals to 1 if the cell at the height h_i' is free or active (i.e., the growth may occur at the next step) or 0 if the cell is blocked or inactive. F_i is the interface activity function.

Averaging over the lattice, taking $\delta t\to 0$, the evolution equation for the mean height is

$$\frac{dh}{dt}=\langle W_iF_i\rangle+\langle 1-W_i\rangle, \tag{A2}$$

and the evolution equation for the square interface width is

$$\frac{dw^2}{dt}=2\langle(h_i-\langle h_i\rangle)W_iF_i\rangle+2\langle[\min(h_{i-1},h_{i+1})-\langle h_i\rangle](1-W_i)\rangle. \tag{A3}$$

The first terms of both equations have been identified as the substratum growth contributions, and the second terms as the diffusion growth contribution.

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